Indian Statistical Institute
Semestral Examination
Differential Geometry - MMath I

Max Marks: 60 Time: 180 minutes.

Answer all questions.

(1) (a) Let $f : U \longrightarrow \mathbb{R}$ be a smooth function ($U \subseteq \mathbb{R}^{n+1}$ open) and let $\alpha : I \longrightarrow U$ be an integral curve of ∇f. Prove the following :
(i) $\frac{d}{dt}(f \circ \alpha)(t) = ||\nabla f(\alpha(t)||^2$ for all $t \in I$,
(ii) If $\beta : J \longrightarrow U$ is such that $\beta(s_0) = \alpha(t_0)$ for some $s_0 \in J$, $t_0 \in I$ and $||\dot{\beta}(s_0)|| = ||\dot{\alpha}(t_0)||$, then

$$\frac{d}{dt}(f \circ \alpha)(t_0) \geq \frac{d}{dt}(f \circ \beta)(s_0).$$

[3+4]

(b) Show that $SL_2(\mathbb{R})$, the set of 2×2 real matrices with determinant equal to 1, is a 3-surface in $\mathbb{R}^4$. Describe the tangent spaces $SL_2(\mathbb{R})_p$, where $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. [8]

(2) (a) Show that a parametrized curve α in the unit sphere $x_1^2 + \cdots + x_{n+1}^2 = 1$ is a geodesic if and only if it is of the form $\alpha(t) = (\cos at)e_1 + (\sin at)e_2$ where e_1, e_2 are a pair of orthogonal unit vectors in $\mathbb{R}^{n+1}$. [7]

(b) Let S be a n-plane in $\mathbb{R}^{n+1}$. Show that parallel transport in S is independent of the path. [8]

(3) (a) A smooth tangent vector field $\mathbb{X}$ on an n-surface S is said to be a geodesic vector field if all its integral curves are geodesics of S. Show that a smooth tangent vector field $\mathbb{X}$ on S is a geodesic vector field if and only if the covariant derivative

$$D_{\mathbb{X}(p)}\mathbb{X} = 0$$

for all $p \in S$. [8]

(b) Find a global parametrization of the plane curve $x_2 - ax_1^2 = c$, $a \neq 0$ and compute its curvature κ. [7]

(4) (a) Let φ be the parametrized surface in $\mathbb{R}^3$ given by

$$\varphi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi).$$

Compute the principal curvatures and the Gaussian curvature of the surface. [10]

(b) Let $g : U \longrightarrow \mathbb{R}$ be a smooth function defined on an open set $U \subseteq \mathbb{R}^n$. Define $\varphi : U \longrightarrow R^{n+1}$ by

$$\varphi(u_1, \ldots, u_n) = (u_1, \ldots, u_n, g(u_1, \ldots, u_n)).$$

Show that

$$V(\varphi) = \int_U \left(1 + \Sigma(\partial g/\partial u_i)^2\right)^{1/2}.$$

[5]